

Free boundary problems of the competition model with sign-changing coefficients in heterogeneous time-periodic environment¹

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Abstract. In this paper we consider two kinds of free boundary problems for the diffusive competition model in the heterogeneous time-periodic environment, in which the variable intrinsic growth rates of invasive and native species may change signs and be “very negative” in a “suitable large region” (see the conditions **(A)** and **(H1)**). The main purpose is to understand the dynamical behavior of the two competing species spreading via a free boundary. We study the spreading-vanishing dichotomy, long time behavior of solution, sharp criteria for spreading and vanishing, and estimates of the asymptotic spreading speed of the free boundary. Moreover, the existence of positive solutions to a T -periodic boundary value problem in half line, associated with our free boundary problems, is obtained.

Keywords: Diffusive competition model; heterogeneous time-periodic environment; Sign-changing coefficients; Free boundary problem; Spreading and vanishing.

AMS subject classifications (2000): 35K51, 35R35, 92B05, 35B40.

1 Introduction

In the natural world, the following phenomenon will happen constantly:

- There is one kind of native species in an area (initial habitat). At some time (initial time), a new or invasive species (competitor) enters this district.

A typical mathematical model describing the interaction (competition) between invasive and native species is the following competitive model

$$\begin{cases} u_t - d_1 u_{xx} = u(a(t, x) - c(t, x)u - k(t, x)v), \\ v_t - d_2 v_{xx} = v(b(t, x) - m(t, x)v - h(t, x)u), \end{cases} \quad (1.1)$$

where $u(t, x)$ and $v(t, x)$ represent the population densities of the invasive and native species, respectively; $d_1, d_2 > 0$ are their diffusion (dispersal) rates; $a(t, x)$, $b(t, x)$ denote their intrinsic growth rates; $c(t, x)$, $m(t, x)$ are the intraspecific and $k(t, x)$, $h(t, x)$ the interspecific competition rates. The system (1.1), as a model describing the spreading, persistence and extinction of two competing species in the heterogeneous environment, has received an astonishing amount of attention, please refer to [2, 3, 4, 16, 20, 24] for example. When the functions a, b, c, m, k and h are positive constants, to describe the invasion and spreading phenomenon, there have been many interesting studies on positive traveling waves and asymptotic spreading speed of (1.1), see, for example [13, 21, 39] and the references cited therein.

In the natural world, for most animals and plants, their birth and death rates will change with seasons, so the intrinsic growth rates $a(t, x), b(t, x)$ and then the intraspecific competition rates $c(t, x), m(t, x)$ and interspecific competition rates $k(t, x), h(t, x)$ should be time-periodic functions.

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Especially, in the winter of severe cold and cold zones, animals cannot capture enough food to feed upon and will not breed, seeds cannot germinate and buds cannot grow above ground, so their birth rates are zero. In the meantime, their death rate will be greater. Therefore, in some periods and some areas, the intrinsic growth rates $a(t, x)$ and $b(t, x)$ may be negative.

In general, the invasive and native species will have a tendency to emigrate from the boundary to obtain their new habitats, i.e., they will move outward along the unknown curve (free boundary) as time increases. In order to simplify the mathematics, in this paper we only consider the one dimensional case and assume that the left boundary is fixed: $x = 0$. Moreover, we take $c(t, x)$, $m(t, x)$, $k(t, x)$ and $h(t, x)$ are positive constants, and by the suitable rescaling we may think that $c(t, x) = m(t, x) = 1$. It should be emphasized that for the higher dimensional and radially symmetric case, when c , m , k and h are functions of (t, x) and have positive lower and upper bounds, the methods used in this paper are still valid and related results remains hold.

If, at the initial time, the range occupied by native species is not very large and the invasive species has a wide distribution in this area, we can use the following free boundary problem to model the above phenomenon:

$$\begin{cases} u_t - d_1 u_{xx} = u(a(t, x) - u - kv), & t > 0, \quad 0 < x < s(t), \\ v_t - d_2 v_{xx} = v(b(t, x) - v - hu), & t > 0, \quad 0 < x < s(t), \\ B_1[u] = B_2[v] = 0, & t > 0, \quad x = 0, \\ u = v = 0, \quad s'(t) = -\mu(u_x + \rho v_x), & t > 0, \quad x = s(t), \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), & 0 \leq x \leq s_0, \\ s(0) = s_0. \end{cases} \quad (1.2)$$

If, at the initial time, the distribution range of native species is very large (it can be considered to be half line) and the invasive species has a local distribution in this area, we shall use the following free boundary problem to describe the above phenomenon:

$$\begin{cases} u_t - d_1 u_{xx} = u(a(t, x) - u - kv), & t > 0, \quad 0 < x < s(t), \\ u(t, x) \equiv 0, & t > 0, \quad x \geq s(t), \\ v_t - d_2 v_{xx} = v(b(t, x) - v - hu), & t > 0, \quad 0 < x < \infty, \\ B_1[u] = B_2[v] = 0, & t > 0, \quad x = 0, \\ u = 0, \quad s'(t) = -\mu u_x, & t > 0, \quad x = s(t), \\ u(0, x) = u_0(x), \quad 0 \leq x \leq s_0; \quad v(0, x) = v_0(x), \quad x \geq 0, \\ s(0) = s_0. \end{cases} \quad (1.3)$$

In the above two problems, $B_1[u] = \alpha_1 u - \beta_1 u_x$, $B_2[v] = \alpha_2 v - \beta_2 v_x$, α_i, β_i are nonnegative constants and satisfy $\alpha_i + \beta_i = 1$; $x = s(t)$ is the free boundary to be determined; positive constant s_0 is the initial boundary or survival range; positive constants μ and $\mu\rho$, the expansion capacities, are the ratios of the expansion speed of the free boundary relative to population gradients at the expanding front, those describe the abilities of species to transmit and dispersal in the new habitat and can also be considered as the “moving parameters”.

Throughout this paper, we assume that

(H) Functions $a, b \in (C^{\frac{\nu}{2}, \nu} \cap L^\infty)([0, \infty) \times [0, \infty))$ for some $\nu \in (0, 1)$, and are T -periodic in time t for some $T > 0$ and positive somewhere in $[0, T] \times [0, \infty)$;

and the initial functions $u_0(x), v_0(x)$ satisfy

- $u_0, v_0 \in C^2([0, s_0])$, $u_0, v_0 > 0$ in $(0, s_0)$, $B_1[u_0](0) = u_0(s_0) = 0$ and $B_2[v_0](0) = v_0(s_0) = 0$ for the problem (1.2);
- $u_0 \in C^2([0, s_0])$, $v_0 \in C^2([0, \infty))$, $u_0 > 0$ in $(0, s_0)$, $v_0 > 0$ in $(0, \infty)$, $B_1[u_0](0) = u_0(s_0) = 0$ and $B_2[v_0](0) = 0$ for the problem (1.3).

Some problems associated with (1.2) and (1.3) have been studied recently. When the functions a, b are positive constants, the problem (1.2) has been studied by Guo & Wu [14] and Wang & Zhao [36] for the case that $\alpha_1 = \alpha_2 = 0$, and by Wang & Zhao [36] for the case that $\beta_1 = \beta_2 = 0$. When the functions a, b are positive constants, or $a, b \in (C^1 \cap L^\infty)([0, \infty))$ are independent of time t and strictly positive, the problem (1.3) with $\alpha_1 = \alpha_2 = 0$ has been studied by Du & Lin [10] and Wang & Zhang [29] for the higher dimension and radially symmetric case. The complete conclusions about spreading-vanishing dichotomy, sharp criteria for spreading and vanishing, long time behaviour of (u, v) and asymptotic spreading speed of the free boundary have been obtained in [14, 36, 10]. Some free boundary problems of diffusive prey-predator model with positive constant coefficients has been studied by Wang, Zhang & Zhao [30, 32, 34, 35, 38].

In the absence of a native species, namely $v \equiv 0$, both problems (1.2) and (1.3) reduce to the following diffusive logistic problem with a free boundary

$$\begin{cases} u_t - d_1 u_{xx} = u(a(t, x) - u), & t > 0, \quad 0 < x < s(t), \\ B_1[u](t, 0) = 0, \quad u(t, s(t)) = 0, & t > 0, \\ s'(t) = -\mu u_x(t, s(t)), & t > 0, \\ s(0) = s_0, \quad u(0, x) = u_0(x), & 0 \leq x \leq s_0, \end{cases} \quad (1.4)$$

which may be used to describe the spreading of a new or invasive species and has been studied by the author in [33] recently. When the function a has positive lower and upper bounds, i.e., there exist positive constants κ_1, κ_2 such that $\kappa_1 \leq a(t, x) \leq \kappa_2$, Du, Guo & Peng in [7] have studied the problem (1.4) with $\alpha_1 = 0$ for the higher dimension and radially symmetric case. The spreading-vanishing dichotomy, sharp criteria for spreading and vanishing and asymptotic spreading speed of the free boundary have been obtained in [7, 33].

When $a = a(x)$ is independent of the time t and changes sign, the problem (1.4) was studied by Zhou & Xiao [40] and Wang [31]. When $a = a(x)$ has positive lower and upper bounds, some similar problems to (1.4) has been studied systematically. When a is a positive constant, the problem (1.4) was investigated earlier by Du & Lin [9] for $\alpha_1 = 0$ and by Kaneko & Yamada [18] for $\beta_1 = 0$. Du, Guo & Liang [5, 8] discussed the higher dimensional and radially symmetric case ($\alpha_1 = 0$). The non-radial case in higher dimensions was treated by Du & Guo [6]. Peng & Zhao [26] studied a free boundary problem of the diffusive logistic model with seasonal succession. They considered that the species does not migrate and stays in a hibernating status in bad season. The evolution of the species obeys Malthusian's equation $u_t = -\delta u$ in bad season, and obeys the diffusive logistic equation with positive constant coefficients in good season. Instead of $u(a - bu)$ by a general function $f(u)$, Du, Matsuzawa & Zhou [12], Kaneko [17] and Du & Lou [11] investigated the corresponding free boundary problems.

The main aim of this paper is to study the dynamical properties of (1.2) and (1.3). We first state the global existence, uniqueness, regularity and estimate of solution (u, v, s) .

Theorem 1.1 *The problem (1.2) has a unique global solution (u, v, s) in time and satisfies*

$$u, v \in C^{1+\frac{\nu}{2}, 2+\nu}(D_\infty), \quad s \in C^{1+\frac{1+\nu}{2}}((0, \infty)), \quad (1.5)$$

where $D_\infty = \{t > 0, x \in [0, s(t)]\}$. Furthermore, there exist constants $M = M(\|a, b, u_0, v_0\|_\infty) > 0$ and $C = C(\mu, \|a, b, u_0, v_0\|_\infty) > 0$, such that

$$0 < u(t, x), v(t, x) \leq M, \quad 0 < s'(t) \leq \mu M, \quad \forall t > 0, 0 < x < s(t) \quad (1.6)$$

and

$$\begin{cases} \|u(t, \cdot), v(t, \cdot)\|_{C^1[0, s(t)]} \leq C, & \forall t \geq 1, \\ \|s'\|_{C^{\nu/2}([n+1, n+3])} \leq C, & \forall n \geq 0. \end{cases} \quad (1.7)$$

Theorem 1.2 *The problem (1.3) has a unique global solution (u, v, s) in time and*

$$u \in C^{1+\frac{\nu}{2}, 2+\nu}(D_\infty), \quad v \in C^{1+\frac{\nu}{2}, 2+\nu}((0, \infty) \times [0, \infty)), \quad s \in C^{1+\frac{1+\nu}{2}}((0, \infty)).$$

Moreover, (u, v, s) satisfies the estimates

$$0 < u(t, x), v(t, x) \leq M, \quad 0 < s'(t) \leq \mu M, \quad \forall t, x > 0$$

and

$$\begin{cases} \|u(t, \cdot)\|_{C^1([0, s(t)])}, \|v(t, \cdot)\|_{C^1([0, \infty))} \leq C, & \forall t \geq 1, \\ \|s'\|_{C^{\nu/2}([n+1, n+3])} \leq C, & \forall n \geq 0, \end{cases} \quad (1.8)$$

where D_∞ , M and C are same as those of Theorem 1.1.

Proofs of Theorems 1.1 and 1.2 are essentially parallel to that of [10, 14, 33]. For the global existence and uniqueness of (u, v, s) , please refer to [10, Theorem 2.1] and [14, Theorem 1]; for the regularities and estimates of (u, v, s) , please refer to [33, Theorem 2.1]. The details are omitted here. We remark that the uniform estimates (1.7) and (1.8) allow us to deduce that $s'(t) \rightarrow 0$ when $s_\infty < \infty$ and play a key role for determining the vanishing phenomenon.

It follows from Theorems 1.1 and 1.2 that $s(t)$ is monotonically increasing. There exists $s_\infty \in (0, \infty]$ such that $\lim_{t \rightarrow \infty} s(t) = s_\infty$.

In order to study the long time behavior of solution and spreading phenomenon, in Section 2 we investigate a stationary problem: the T -periodic boundary value problem in half line associated with the free boundary problems (1.2) and (1.3). As most properties of (1.2) and (1.3) are similar, we first deal with the problem (1.2) meticulously in Sections 3 and 4, and briefly discuss the problem (1.3) in Section 5. In Section 3, we shall derive the spreading-vanishing dichotomy of (1.2):

Either

(i) spreading: $s_\infty = \infty$ and

$$\begin{aligned} U_*(t, x) &\leq \liminf_{n \rightarrow \infty} u(t + nT, x), & \limsup_{n \rightarrow \infty} u(t + nT, x) &\leq U^*(t, x), \\ V_*(t, x) &\leq \liminf_{n \rightarrow \infty} v(t + nT, x), & \limsup_{n \rightarrow \infty} v(t + nT, x) &\leq V^*(t, x), \end{aligned}$$

uniformly in $[0, T] \times [0, L]$ for any $L > 0$, where (U^*, V_*) and (U_*, V^*) are positive T -periodic solutions of (2.1) which will be given in Theorem 2.1;

or

(ii) vanishing: $s_\infty < \infty$ and $\lim_{t \rightarrow \infty} \|u(t, \cdot), v(t, \cdot)\|_{C([0, s(t)])} = 0$.

In Section 4, the criteria for spreading and vanishing of the problem (1.2) will be established.

2 Positive solutions of a T -periodic boundary value problem in half line

To discuss the long time behavior of solution and spreading phenomenon, we should study the following T -periodic boundary value problem in half line

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - U - kV), & 0 < t < T, \ 0 < x < \infty, \\ V_t - d_2 V_{xx} = V(b(t, x) - V - hU), & 0 < t < T, \ 0 < x < \infty, \\ B_1[U](t, 0) = B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ U(0, x) = U(T, x), \ V(0, x) = V(T, x), & 0 \leq x < \infty. \end{cases} \quad (2.1)$$

2.1 Preliminaries

In order to facilitate applications, in this subsection we state some known results: the comparison principle, properties of the principal eigenvalue of the T -periodic eigenvalue problem and conclusions of the diffusive logistic equation in the heterogeneous time-periodic environment.

Lemma 2.1 (*Comparison principle*) *Let $\tau, \ell > 0$. Assume that*

$$\overline{w}, \underline{w}, \overline{z}, \underline{z} \in C([0, \tau] \times [0, \ell]) \cap C^{0,1}([0, \tau] \times [0, \ell]) \cap C^{1,2}((0, \tau] \times (0, \ell)),$$

and are nonnegative functions. If $(\overline{w}, \underline{z})$ and $(\underline{w}, \overline{z})$ satisfy

$$\begin{cases} \overline{w}_t - d_1 \overline{w}_{xx} \geq \overline{w}(a(t, x) - \overline{w} - k\underline{z}), & 0 < t \leq \tau, \ 0 < x < \ell, \\ \underline{z}_t - d_2 \underline{z}_{xx} \leq \underline{z}(b(t, x) - \underline{z} - h\overline{w}), & 0 < t \leq \tau, \ 0 < x < \ell, \\ \underline{w}_t - d_1 \underline{w}_{xx} \leq \underline{w}(a(t, x) - \underline{w} - k\overline{z}), & 0 < t \leq \tau, \ 0 < x < \ell, \\ \overline{z}_t - d_2 \overline{z}_{xx} \geq \overline{z}(b(t, x) - \overline{z} - h\underline{w}), & 0 < t \leq \tau, \ 0 < x < \ell \end{cases}$$

and

$$\begin{cases} B_1[\overline{w}](t, 0) \geq B_2[\underline{z}](t, 0), & 0 \leq t \leq \tau, \\ B_1[\underline{w}](t, 0) \leq B_2[\overline{z}](t, 0), & 0 \leq t \leq \tau, \\ \overline{w}(t, \ell) \geq \underline{w}(t, \ell), \ \underline{z}(t, \ell) \leq \overline{z}(t, \ell), & 0 \leq t \leq \tau, \\ \overline{w}(0, x) \geq \underline{w}(0, x), \ \underline{z}(0, x) \leq \overline{z}(0, x), & 0 \leq x \leq \ell. \end{cases}$$

Then we have

$$\underline{w} \leq \overline{w}, \ \underline{z} \leq \overline{z} \quad \text{in } [0, \tau] \times [0, \ell].$$

Proof. Note that the function pair $(u(a(t, x) - u - kv), v(b(t, x) - v - hu))$ of (u, v) is *quasi-monotone nonincreasing* in $u, v \geq 0$. The desired result can be deduced by the comparison principle for parabolic systems (see [27] or [37, section 4.2.2]). We omit the details here. \square

In the following we assume that $c, q \in (C^{\frac{\nu}{2}, \nu} \cap L^\infty)([0, \infty) \times [0, \infty))$ and are T -periodic functions in time t . Moreover, there exist positive constants $\underline{q}, \overline{q}$ such that $\underline{q} \leq q(t, x) \leq \overline{q}$ for all $t, x \geq 0$. Define $B[u] = \alpha u - \beta u_x$, where α and β are non-negative constants and satisfy $\alpha + \beta = 1$.

Lemma 2.2 ([33, Lemma 3.2]) *Let $d, \ell > 0$, $\overline{z}, \underline{z} \in C^{1,2}([0, T] \times (0, \ell)) \cap C^{0,1}([0, T] \times [0, \ell])$. If $(\overline{z}, \underline{z})$ satisfies*

$$\begin{cases} \overline{z}_t - d\overline{z}_{xx} \geq c(t, x)\overline{z} - q(t, x)\overline{z}^2, & 0 \leq t \leq T, \ 0 < x < \ell, \\ \underline{z}_t - d\underline{z}_{xx} \leq c(t, x)\underline{z} - q(t, x)\underline{z}^2, & 0 \leq t \leq T, \ 0 < x < \ell, \\ B[\overline{z}](t, 0) \geq B[\underline{z}](t, 0), \ \overline{z}(t, \ell) \geq \underline{z}(t, \ell), & 0 \leq t \leq T, \\ \overline{z}(0, x) = \overline{z}(T, x), \ \underline{z}(0, x) = \underline{z}(T, x), & 0 \leq x \leq \ell. \end{cases}$$

Then $\bar{z} \geq \underline{z}$ in $[0, T] \times [0, \ell]$.

For any given $d, \ell > 0$, let $\lambda_1(\ell; d, c)$ be the principal eigenvalue of the following T -periodic eigenvalue problem

$$\begin{cases} \phi_t - d\phi_{xx} - c(t, x)\phi = \lambda\phi, & 0 \leq t \leq T, \ 0 < x < \ell, \\ B[\phi](t, 0) = 0, \ \phi(t, \ell) = 0, & 0 \leq t \leq T, \\ \phi(0, x) = \phi(T, x), & 0 \leq x \leq \ell. \end{cases} \quad (2.2)$$

Proposition 2.1 ([33, Proposition 3.1]) *The principal eigenvalue $\lambda_1(\ell; d, c)$ is continuous with respect to ℓ, d and c , and strictly decreasing in c and ℓ . Moreover, $\lim_{\ell \rightarrow 0^+} \lambda_1(\ell; d, c) = \infty$ and $\lim_{d \rightarrow \infty} \lambda_1(\ell; d, c) = \infty$.*

Proposition 2.2 ([33, Proposition 3.2]) *Assume that the function $c(t, x)$ satisfies*

(A) *There exist $\varsigma > 0$, $-2 < r \leq 0$, $m > 1$ and x_n satisfying $x_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $c(t, x) \geq \varsigma x^r$ in $[0, T] \times [x_n, mx_n]$.*

Then for any given $d > 0$, there exists a unique $\ell_0 = \ell_0(d) > 0$ such that $\lambda_1(\ell_0; d, c) = 0$. Hence, $\lambda_1(\ell; d, c) < 0$ for all $\ell > \ell_0$.

Consider the following T -periodic boundary value problem of the diffusive logistic equation in a bounded interval $(0, \ell)$:

$$\begin{cases} W_t - dW_{xx} = W(c(t, x) - q(t, x)W), & 0 \leq t \leq T, \ 0 < x < \ell, \\ B[W](t, 0) = 0, \ W(t, \ell) = K, & 0 \leq t \leq T, \\ W(0, x) = W(T, x), & 0 \leq x \leq \ell. \end{cases} \quad (2.3)$$

Lemma 2.3 ([33, Lemma 3.3]) *Assume that $c(t, x)$ satisfies the condition (A). Then, for any given $\ell > \ell_0$ and $K \geq \|c\|_\infty / \underline{q}$, the problem (2.3) has a unique positive solution.*

Now, let us consider the following initial-boundary value problem and T -periodic boundary value problem of the diffusive logistic equation in the half line:

$$\begin{cases} w_t - dw_{xx} = w(c(t, x) - w), & t > 0, \ 0 < x < \infty, \\ B[w](t, 0) = 0, & t > 0, \\ w(0, x) = w_0(x), & 0 \leq x < \infty \end{cases} \quad (2.4)$$

and

$$\begin{cases} W_t - dW_{xx} = W(c(t, x) - W), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B[W](t, 0) = 0, & 0 \leq t \leq T, \\ W(0, x) = W(T, x), & 0 \leq x < \infty, \end{cases} \quad (2.5)$$

where $w_0(x)$ is a bounded nontrivial and nonnegative continuous function. For the convenience to write, we first a definition.

Definition 2.1 Let r be a constant and satisfy $-2 < r \leq 0$, $c \in (C^{\frac{\nu}{2}, \nu} \cap L^\infty)([0, \infty) \times [0, \infty))$ be a T -periodic function in time t . We call that c belongs to the class $\mathcal{C}_r(T)$ if there exist two positive T -periodic functions $c_\infty(t), c^\infty(t) \in C^{\nu/2}([0, T])$, such that

$$c_\infty(t) \leq \liminf_{x \rightarrow \infty} \frac{c(t, x)}{x^r}, \quad \limsup_{x \rightarrow \infty} \frac{c(t, x)}{x^r} \leq c^\infty(t) \quad \text{uniformly in } [0, T].$$

It is easy to see that if $c \in \mathcal{C}_r(T)$ for some $-2 < r \leq 0$, then c satisfies the condition **(A)**.

Proposition 2.3 Assume that $c \in \mathcal{C}_r(T)$. Then the T -periodic boundary value problem (2.5) has a unique positive solution $W \in C^{1+\frac{\nu}{2}, 2+\nu}([0, T] \times [0, \infty)) \cap \mathcal{C}_r(T)$, and satisfies

$$\min_{[0, T]} c_\infty(t) \leq \liminf_{x \rightarrow \infty} \frac{W(t, x)}{x^r}, \quad \limsup_{x \rightarrow \infty} \frac{W(t, x)}{x^r} \leq \max_{[0, T]} c^\infty(t) \quad (2.6)$$

uniformly in $[0, T]$. Moreover, the solution w of (2.4) satisfies

$$\lim_{n \rightarrow \infty} w(t + nT) = W(t, x) \quad \text{locally uniformly in } [0, T] \times [0, \infty). \quad (2.7)$$

Especially, if $r = 0$ then

$$w_\infty(t) \leq \liminf_{x \rightarrow \infty} W(t, x), \quad \limsup_{x \rightarrow \infty} W(t, x) \leq w^\infty(t) \quad (2.8)$$

uniformly in $[0, T]$, where $w_\infty(t)$ and $w^\infty(t)$ are, respectively, the unique positive solutions of the following T -periodic ordinary differential problems:

$$w'(t) = w(c_\infty(t) - w), \quad w(0) = w(T),$$

and

$$w'(t) = w(c^\infty(t) - w), \quad w(0) = w(T).$$

Proof. Existence and uniqueness of W together with the estimate (2.6) is just Theorem 4.2 of [33]. The limit (2.7) is given by Theorem 4.3 of [33]. The estimate (2.8) can be proved by the similar method to that of [25, Theorem 1.4]. We omit the details here. \square

Proposition 2.4 (Comparison principle) Assume that $-2 < r_i \leq 0$ and $c_i \in \mathcal{C}_{r_i}(T)$, $i = 1, 2$. Let $W_i \in C^{1+\frac{\nu}{2}, 2+\nu}([0, T] \times [0, \infty))$ be the unique positive solution of (2.5) with $c = c_i$. If $c_1 \leq c_2$, then $W_1 \leq W_2$ in $[0, T] \times [0, \infty)$.

Proof. The existence and uniqueness of W_i is guaranteed by Proposition 2.3. Since $c_i \in \mathcal{C}_{r_i}(T)$, we have that c_i satisfies the condition **(A)**. In view of Proposition 2.2, there exists $\ell_0 = \ell_0(d) \gg 1$ such that $\lambda_1(\ell; d, c_i) < 0$ for $\ell > \ell_0$ and $i = 1, 2$.

For any given $\ell > \ell_0$. Utilizing Theorem 28.1 of [15], the problem

$$\begin{cases} W_t - dW_{xx} = W(c_i(t, x) - W), & 0 \leq t \leq T, \ 0 < x < \ell, \\ B[W](t, 0) = 0, \ W(\ell, t) = 0, & 0 \leq t \leq T, \\ W(0, x) = W(T, x), & 0 \leq x \leq \ell \end{cases}$$

has a unique positive T -periodic solution $W_{i\ell}(t, x)$. Since $c_1 \leq c_2$, by Lemma 2.2,

$$W_{1\ell}(t, x) \leq W_{2\ell}(t, x) \quad \text{in } [0, T] \times [0, \ell]. \quad (2.9)$$

Obviously, $W_{i\ell} \leq \|c\|_\infty$ by the maximum principle, and $W_{i\ell}$ is increasing in ℓ by Lemma 2.2. Remember $W_i(t, x)$ is the unique positive solution of (2.5) with $c = c_i$. Make use of the regularity theory for parabolic equations and compact argument, it can be proved that, for any given $L > 0$, $W_{i\ell} \rightarrow W_i$ in $C^{1,2}([0, T] \times [0, L])$ as $\ell \rightarrow \infty$, $i = 1, 2$. These facts combined with (2.9) allow us to derive $W_1 \leq W_2$. The proof is complete. \square

2.2 Existence and properties of positive solutions to (2.1)

We first state a condition.

(H1) The functions $a, b \in \mathcal{C}_r(T)$ for some $-2 < r \leq 0$. That is, there exist positive T -periodic functions $a_\infty(t), b_\infty(t), a^\infty(t), b^\infty(t) \in C^{\nu/2}([0, T])$, such that

$$\begin{cases} a_\infty(t) \leq \liminf_{x \rightarrow \infty} \frac{a(t, x)}{x^r}, & \limsup_{x \rightarrow \infty} \frac{a(t, x)}{x^r} \leq a^\infty(t), \\ b_\infty(t) \leq \liminf_{x \rightarrow \infty} \frac{b(t, x)}{x^r}, & \limsup_{x \rightarrow \infty} \frac{b(t, x)}{x^r} \leq b^\infty(t) \end{cases} \quad (2.10)$$

uniformly in $[0, T]$.

When the condition **(H1)** holds, we define

$$\underline{a}_\infty = \min_{[0, T]} a_\infty(t), \quad \bar{a}^\infty = \max_{[0, T]} a^\infty(t), \quad \underline{b}_\infty = \min_{[0, T]} b_\infty(t), \quad \bar{b}^\infty = \max_{[0, T]} b^\infty(t).$$

Theorem 2.1 *Under the condition (H1), we assume further that*

$$\underline{b}_\infty > h\bar{a}^\infty, \quad \underline{a}_\infty > k\bar{b}^\infty. \quad (2.11)$$

Then there exist four positive T -periodic functions $U^, U_*, V^*, V_* \in C^{1+\frac{\nu}{2}, 2+\nu}([0, T] \times [0, \infty))$, such that both (U^*, V_*) and (U_*, V^*) are positive solutions of (2.1). Moreover, any positive solution (U, V) of (2.1) satisfies*

$$U_* \leq U \leq U^*, \quad V_* \leq V \leq V^* \quad \text{in } [0, T] \times [0, \infty). \quad (2.12)$$

Epecially, if $r = 0$ in (H1), then any positive solution (U, V) of (2.1) satisfies

$$\begin{cases} w_1(t) \leq \liminf_{x \rightarrow \infty} U(t, x), & \limsup_{x \rightarrow \infty} U(t, x) \leq w_2(t), \\ z_1(t) \leq \liminf_{x \rightarrow \infty} V(t, x), & \limsup_{x \rightarrow \infty} V(t, x) \leq z_2(t) \end{cases} \quad (2.13)$$

uniformly in $[0, T]$, where $w_2(t), z_1(t), z_2(t)$ and $w_1(t)$ are the unique positive solutions of the following T -periodic ordinary differential problems

$$\begin{aligned} w_2'(t) &= w_2(a^\infty(t) - w_2), & w_2(0) &= w_2(T), \\ z_1'(t) &= z_1(b_\infty(t) - hw_2(t) - z_1), & z_1(0) &= z_1(T), \end{aligned} \quad (2.14)$$

$$z_2'(t) = z_2(b^\infty(t) - z_2), \quad z_2(0) = z_2(T), \quad (2.15)$$

and

$$w_1'(t) = w_1(a_\infty(t) - kz_2(t) - w_1), \quad w_1(0) = w_1(T),$$

respectively.

Remark 2.1 *The condition (2.11) is equivalent to $h < \underline{b}_\infty/\bar{a}^\infty$ and $k < \underline{a}_\infty/\bar{b}^\infty$, which corresponds to the weak competitive case.*

Proof of Theorem 2.1. The approach of this proof can be regarded as the upper and lower solutions method. This proof will be divided into four steps. In the first one, we shall construct four positive T -periodic functions \underline{U} , \underline{V} , \overline{U} and \overline{V} , for which $(\overline{U}, \overline{V})$ and $(\underline{U}, \underline{V})$ can be regarded as *coupled ordered upper and lower solutions* of (2.1). In step 2, by use of the functions \underline{U} , \underline{V} , \overline{U} and \overline{V} , we prove the existences of U^* , U_* , V^* and V_* . Proofs of (2.12) and (2.13) will be given in the third and fourth steps, respectively.

Step 1. The construction of \underline{U} , \underline{V} , \overline{U} and \overline{V} . Since $a \in \mathcal{C}_r(T)$, take advantage of Proposition 2.3 we know that the problem

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - U), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_1[U](t, 0) = 0, & 0 \leq t \leq T, \\ U(0, x) = U(T, x), & 0 \leq x < \infty \end{cases} \quad (2.16)$$

admits a unique positive solution $\overline{U}(t, x) \in \mathcal{C}_r(T)$, and

$$\underline{a}_\infty \leq \liminf_{x \rightarrow \infty} \frac{\overline{U}(t, x)}{x^r}, \quad \limsup_{x \rightarrow \infty} \frac{\overline{U}(t, x)}{x^r} \leq \overline{a}^\infty, \quad (2.17)$$

$$\limsup_{x \rightarrow \infty} \overline{U}(t, x) \leq w_2(t) \quad \text{if } r = 0 \quad (2.18)$$

uniformly in $[0, T]$. Moreover, $\overline{U} \leq \|a\|_\infty$ by the maximum principle. It follows that $b - h\overline{U} \in \mathcal{C}_r(T)$ since $b \in \mathcal{C}_r(T)$ and $\underline{b}_\infty - h\overline{a}^\infty > 0$. Applying Proposition 2.3 again, the problem

$$\begin{cases} V_t - d_2 V_{xx} = V(b(t, x) - h\overline{U}(t, x) - V), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ V(0, x) = V(T, x), & 0 \leq x < \infty \end{cases} \quad (2.19)$$

has a unique positive solution $\underline{V}(t, x) \in \mathcal{C}_r(T)$, and

$$\underline{b}_\infty - h\overline{a}^\infty \leq \liminf_{x \rightarrow \infty} \frac{\underline{V}(t, x)}{x^r}, \quad \limsup_{x \rightarrow \infty} \frac{\underline{V}(t, x)}{x^r} \leq \overline{b}^\infty - h\underline{a}_\infty, \quad (2.20)$$

$$\liminf_{x \rightarrow \infty} \underline{V}(t, x) \geq z_1(t) \quad \text{if } r = 0 \quad (2.21)$$

uniformly in $[0, T]$.

Similarly to the above, the problem

$$\begin{cases} V_t - d_2 V_{xx} = V(b(t, x) - V), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ V(0, x) = V(T, x), & 0 \leq x < \infty \end{cases}$$

admits a unique positive solution $\overline{V} \in \mathcal{C}_r(T)$, and the problem

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - k\overline{V}(t, x) - U), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_1[U](t, 0) = 0, & 0 \leq t \leq T, \\ U(0, x) = U(T, x), & 0 \leq x < \infty \end{cases}$$

has a unique positive solution $\underline{U} \in \mathcal{C}_r(T)$. Moreover,

$$w_1(t) \leq \liminf_{x \rightarrow \infty} \underline{U}(t, x), \quad \limsup_{x \rightarrow \infty} \overline{V}(t, x) \leq z_2(t) \quad \text{if } r = 0 \quad (2.22)$$

uniformly in $[0, T]$.

In addition, by Proposition 2.4 we have $\underline{U}(t, x) \leq \overline{U}(t, x)$ and $\underline{V}(t, x) \leq \overline{V}(t, x)$ in $[0, T] \times [0, \infty)$, and hence in $[0, \infty) \times [0, \infty)$ as they are T -periodic functions in time t .

Step 2. In this step we use the functions \underline{U} , \underline{V} , \overline{U} and \overline{V} obtained in the above step to construct U^* , U_* , V^* and V_* , and prove that both (U^*, V_*) and (U_*, V^*) are positive solutions of (2.1). Such a process is probably well known. For completeness, we shall provide the details.

Let $\ell > 0$ and (w_ℓ, z_ℓ) be the unique positive solution of the initial-boundary value problem:

$$\begin{cases} w_t - d_1 w_{xx} = w(a(t, x) - w - kz), & t > 0, 0 < x < \ell, \\ z_t - d_2 z_{xx} = z(b(t, x) - z - hw), & t > 0, 0 < x < \ell, \\ B_1[w](t, 0) = B_2[z](t, 0) = 0, & t \geq 0, \\ w(t, \ell) = \overline{U}(t, \ell), \quad z(t, \ell) = \underline{V}(t, \ell), & t \geq 0, \\ w(0, x) = \overline{U}(0, x), \quad z(0, x) = \underline{V}(0, x), & 0 \leq x \leq \ell. \end{cases} \quad (2.23)$$

Since \overline{U} and \underline{V} are positive functions and satisfy

$$\begin{cases} \overline{U}_t - d_1 \overline{U}_{xx} > \overline{U}(a(t, x) - \overline{U} - k\underline{V}), & t > 0, 0 < x < \ell, \\ \underline{V}_t - d_2 \underline{V}_{xx} = \underline{V}(b(t, x) - \underline{V} - h\overline{U}), & t > 0, 0 < x < \ell, \\ B_1[\overline{U}](t, 0) = B_2[\underline{V}](t, 0) = 0, & t \geq 0, \end{cases}$$

one can use Lemma 2.1 to deduce $w_\ell \leq \overline{U}$, $z_\ell \geq \underline{V}$ in $[0, \infty) \times [0, \ell]$. For the non-negative integer n , we define

$$w_\ell^n(t, x) = w_\ell(t + nT, x), \quad z_\ell^n(t, x) = z_\ell(t + nT, x).$$

Note that $a(t, x)$, $b(t, x)$, $\overline{U}(t, x)$ and $\underline{V}(t, x)$ are T -periodic functions in time t , it follows that (w_ℓ^n, z_ℓ^n) satisfies

$$\begin{cases} (w_\ell^n)_t - d_1 (w_\ell^n)_{xx} = w_\ell^n(a(t, x) - w_\ell^n - kz_\ell^n), & 0 < t \leq T, 0 < x < \ell, \\ (z_\ell^n)_t - d_2 (z_\ell^n)_{xx} = z_\ell^n(b(t, x) - z_\ell^n - hw_\ell^n), & 0 < t \leq T, 0 < x < \ell, \\ B_1[w_\ell^n](t, 0) = B_2[z_\ell^n](t, 0) = 0, & 0 \geq t \leq T, \\ w_\ell^n(t, \ell) = \overline{U}(t, \ell), \quad z_\ell^n(t, \ell) = \underline{V}(t, \ell), & 0 \geq t \leq T, \\ w_\ell^n(0, x) = w_\ell(nT, x), \quad z_\ell^n(0, x) = z_\ell(nT, x), & 0 \leq x \leq \ell. \end{cases}$$

Remember

$$\begin{aligned} w_\ell^1(0, x) &= w_\ell(T, x) \leq \overline{U}(T, x) = \overline{U}(0, x) = w_\ell(0, x), \\ z_\ell^1(0, x) &= z_\ell(T, x) \geq \underline{V}(T, x) = \underline{V}(0, x) = z_\ell(0, x), \end{aligned}$$

it is derived by Lemma 2.1 that $w_\ell^1 \leq w_\ell$, $z_\ell^1 \geq z_\ell$ in $[0, T] \times [0, \ell]$. And then

$$w_\ell^2(0, x) = w_\ell^1(T, x) \leq w_\ell(T, x) = w_\ell^1(0, x), \quad z_\ell^2(0, x) = z_\ell^1(T, x) \geq z_\ell(T, x) = z_\ell^1(0, x).$$

Apply Lemma 2.1 once again, we have $w_\ell^2 \leq w_\ell^1$, $z_\ell^2 \geq z_\ell^1$ in $[0, T] \times [0, \ell]$. Utilizing the inductive method we can show that w_ℓ^n and z_ℓ^n are, respectively, decreasing and increasing in n . So, there exist two non-negative functions \overline{U}_ℓ , \underline{V}_ℓ such that $w_\ell^n \rightarrow \overline{U}_\ell$, $z_\ell^n \rightarrow \underline{V}_\ell$ pointwise in $[0, T] \times [0, \ell]$ as $n \rightarrow \infty$. Obviously, $\overline{U}_\ell(0, x) = \overline{U}_\ell(T, x)$, $\underline{V}_\ell(0, x) = \underline{V}_\ell(T, x)$ as $w_\ell^{n+1}(0, x) = w_\ell^n(T, x)$, $z_\ell^{n+1}(0, x) = z_\ell^n(T, x)$. Based on the regularity theory for parabolic equations and compact argument, it can be

proved that there exists a subsequence $\{n_i\}$, such that $w_\ell^{n_i} \rightarrow \overline{U}_\ell$, $z_\ell^{n_i} \rightarrow \underline{V}_\ell$ in $C^{1,2}([0, T] \times [0, \ell])$ as $i \rightarrow \infty$, and $(\overline{U}_\ell, \underline{V}_\ell)$ satisfies the first four equations of (2.23). Therefore, $(\overline{U}_\ell, \underline{V}_\ell)$ satisfies

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - U - kV), & 0 \leq t \leq T, \ 0 < x < \ell, \\ V_t - d_2 V_{xx} = V(b(t, x) - V - hU), & 0 \leq t \leq T, \ 0 < x < \ell, \\ B_1[U](t, 0) = B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ U(t, \ell) = \overline{U}(t, \ell), \ V(t, \ell) = \underline{V}(t, \ell), & 0 \leq t \leq T, \\ U(0, x) = U(T, x), \ V(0, x) = V(T, x), & 0 \leq x \leq \ell. \end{cases} \quad (2.24)$$

Evidently, $\overline{U}_\ell(t, x) > 0$, $\underline{V}_\ell(t, x) > 0$ in $(0, T] \times [0, \ell]$ because $\overline{U}(t, \ell) > 0$, $\underline{V}(t, \ell) > 0$ in $(0, T]$. This shows that $(\overline{U}_\ell, \underline{V}_\ell)$ is a positive solution of (2.24).

Let $(\varphi_\ell, \psi_\ell)$ be the unique positive solution of the following initial-boundary value problem

$$\begin{cases} \varphi_t - d_1 \varphi_{xx} = \varphi(a(t, x) - \varphi - k\psi), & t > 0, \ 0 < x < \ell, \\ \psi_t - d_2 \psi_{xx} = \psi(b(t, x) - \psi - h\varphi), & t > 0, \ 0 < x < \ell, \\ B_1[\varphi](t, 0) = B_2[\psi](t, 0) = 0, & t \geq 0, \\ \varphi(t, \ell) = \underline{U}(t, \ell), \ \psi(t, \ell) = \overline{V}(t, \ell), & t \geq 0, \\ \varphi(0, x) = \underline{U}(0, x), \ \psi(0, x) = \overline{V}(0, x), & 0 \leq x \leq \ell. \end{cases}$$

Since $\underline{U} \leq \overline{U}$, $\underline{V} \leq \overline{V}$ in $[0, \infty) \times [0, \infty)$, we have $w_\ell \geq \varphi_\ell \geq \underline{U}$, $z_\ell \leq \psi_\ell \leq \overline{V}$ in $[0, \infty) \times [0, \ell]$ by Lemma 2.1. Similarly to the above, there exist two positive functions \underline{U}_ℓ , \overline{V}_ℓ , and a subsequence $\{n_i\}$, such that $\varphi_\ell(t + n_i T, x) \rightarrow \underline{U}_\ell(t, x)$, $\psi_\ell(t + n_i T, x) \rightarrow \overline{V}_\ell(t, x)$ in $C^{1,2}([0, T] \times [0, \ell])$ as $i \rightarrow \infty$, and $(\underline{U}_\ell, \overline{V}_\ell)$ solves

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - U - kV), & 0 \leq t \leq T, \ 0 < x < \ell, \\ V_t - d_2 V_{xx} = V(b(t, x) - V - hU), & 0 \leq t \leq T, \ 0 < x < \ell, \\ B_1[U](t, 0) = B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ U(t, \ell) = \underline{U}(t, \ell), \ V(t, \ell) = \overline{V}(t, \ell), & 0 \leq t \leq T, \\ U(0, x) = U(T, x), \ V(0, x) = V(T, x), & 0 \leq x \leq \ell. \end{cases} \quad (2.25)$$

Recall that $\underline{U} \leq \varphi_\ell \leq w_\ell \leq \overline{U}$, $\underline{V} \leq z_\ell \leq \psi_\ell \leq \overline{V}$ in $[0, \infty) \times [0, \ell]$, it is immediately to get

$$\underline{U} \leq \underline{U}_\ell \leq \overline{U}_\ell \leq \overline{U}, \quad \underline{V} \leq \underline{V}_\ell \leq \overline{V}_\ell \leq \overline{V} \quad \text{in } [0, T] \times [0, \ell]$$

for any given $\ell > 0$. By use of the regularity theory for parabolic equations and compact argument, we can show that there exist two subsequences $\{(\overline{U}_{\ell_j}, \underline{V}_{\ell_j})\}$, $\{(\underline{U}_{\ell_j}, \overline{V}_{\ell_j})\}$ and four positive functions U^* , U_* , V^* , V_* , such that, for any $L > 0$,

$$(\overline{U}_{\ell_j}, \underline{V}_{\ell_j}) \rightarrow (U^*, V_*), \quad (\underline{U}_{\ell_j}, \overline{V}_{\ell_j}) \rightarrow (U_*, V^*) \quad \text{in } [C^{1,2}([0, T] \times [0, L])]^2$$

as $j \rightarrow \infty$. Obviously,

$$\underline{U} \leq U_* \leq U^* \leq \overline{U}, \quad \underline{V} \leq V_* \leq V^* \leq \overline{V} \quad \text{in } [0, T] \times [0, \infty). \quad (2.26)$$

Remember (2.24) and (2.25), it is easy to see that both (U^*, V_*) and (U_*, V^*) are positive T -periodic solutions of (2.1).

Step 3. Prove (2.12). Let (U, V) be a positive solution of (2.1). We only prove $U \leq U^*$, $V \geq V_*$ as the proof of $U \geq U_*$, $V \leq V^*$ is similarly.

Because U satisfies

$$\begin{cases} U_t - d_1 U_{xx} < U(a(t, x) - U), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_1[U](t, 0) = 0, & 0 \leq t \leq T, \\ U(0, x) = U(T, x), & 0 \leq x < \infty, \end{cases}$$

we have $U(t, x) \leq \|a\|_\infty$ for all $(t, x) \in [0, T] \times [0, \infty)$. Choose $\ell \gg 1$. Using Lemma 2.3 and [15, Theorem 28.1], respectively, we have that the following problems

$$\begin{cases} \zeta_t - d_1 \zeta_{xx} = \zeta(a(t, x) - \zeta), & 0 \leq t \leq T, \ 0 < x < \ell, \\ B_1[\zeta](t, 0) = 0, \ \zeta(t, \ell) = \|a\|_\infty, & 0 \leq t \leq T, \\ \zeta(0, x) = \zeta(T, x), & 0 \leq x \leq \ell \end{cases}$$

and

$$\begin{cases} \zeta_t - d_1 \zeta_{xx} = \zeta(a(t, x) - \zeta), & 0 \leq t \leq T, \ 0 < x < \ell, \\ B_1[\zeta](t, 0) = 0, \ \zeta(t, \ell) = 0, & 0 \leq t \leq T, \\ \zeta(0, x) = \zeta(T, x), & 0 \leq x \leq \ell \end{cases}$$

have unique positive solutions $\bar{\zeta}_\ell$ and $\underline{\zeta}_\ell$. We can apply Lemma 2.2 to conclude that $U \leq \bar{\zeta}_\ell$ and $\underline{\zeta}_\ell \leq \bar{\zeta}_\ell$ in $[0, T] \times [0, \ell]$, $\bar{\zeta}_\ell$ and $\underline{\zeta}_\ell$ are decreasing and increasing in ℓ , respectively. Similarly to the proof of Proposition 2.4, we have $\lim_{\ell \rightarrow \infty} \bar{\zeta}_\ell = \bar{U}$ in $C^{1,2}([0, T] \times [0, L])$ for any $L > 0$ since \bar{U} is the unique positive solution of (2.16). Therefore, $U \leq \bar{U}$ in $[0, T] \times [0, \infty)$. Similarly, we have $V \geq \underline{V}$ in $[0, T] \times [0, \infty)$. It follows that (U, V) satisfies

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - U - kV), & 0 < t \leq T, \ 0 < x < \ell, \\ V_t - d_2 V_{xx} = V(b(t, x) - V - hU), & 0 < t \leq T, \ 0 < x < \ell, \\ B_1[U](t, 0) = B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ U(t, \ell) \leq \bar{U}(t, \ell), \ V(t, \ell) \geq \underline{V}(t, \ell), & 0 \leq t \leq T, \\ U(0, x) \leq \bar{U}(0, x), \ V(0, x) \geq \underline{V}(0, x), & 0 \leq x \leq \ell. \end{cases} \quad (2.27)$$

Applying Lemma 2.1 to the problems (2.23) and (2.27), we get that

$$U \leq w_\ell, \ V \geq z_\ell \quad \text{in } [0, T] \times [0, \ell]. \quad (2.28)$$

It can be seen from the arguments of step 2 that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} w_{\ell_j}(t + nt, x) = U^*, \quad \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} z_{\ell_j}(t + nt, x) = V_* \quad (2.29)$$

in $C^{1,2}([0, T] \times [0, L])$ for any $L > 0$. According to $U(t + nT, x) = U(t, x)$ and $V(t + nT, x) = V(t, x)$, it is derived from (2.28) and (2.29) that $U \leq U^*$, $V \geq V_*$.

Step 4. When $r = 0$ in **(H1)**. Summarizing (2.18), (2.21) and (2.22) we obtain

$$\begin{aligned} w_1(t) &\leq \liminf_{x \rightarrow \infty} \underline{U}(t, x), \quad \limsup_{x \rightarrow \infty} \bar{U}(t, x) \leq w_2(t), \\ z_1(t) &\leq \liminf_{x \rightarrow \infty} \underline{V}(t, x), \quad \limsup_{x \rightarrow \infty} \bar{V}(t, x) \leq z_2(t) \end{aligned}$$

uniformly in $[0, T]$. Combining these facts with (2.26) and (2.12), we can derive (2.13). The proof is complete. \square

3 Spreading-vanishing dichotomy of the problem (1.2)

We first state a lemma, by which the vanishing phenomenon is immediately obtained. Moreover, this lemma will play an important role in the establishment of criteria for spreading and vanishing.

Lemma 3.1 ([31, Lemma 3.1]) *Let d, μ and B be as above, $C \in \mathbb{R}$. Assume that functions $g \in C^1([0, \infty))$, $\varphi \in C^{\frac{1+\nu}{2}, 1+\nu}([0, \infty) \times [0, g(t)])$ and satisfy $g(t) > 0$, $\varphi(t, x) > 0$ for $t \geq 0$ and $0 < x < g(t)$. We further suppose that $\lim_{t \rightarrow \infty} g(t) < \infty$, $\lim_{t \rightarrow \infty} g'(t) = 0$ and there exists a constant $K > 0$ such that $\|\varphi(t, \cdot)\|_{C^1[0, g(t)]} \leq K$ for $t > 1$. If (φ, g) satisfies*

$$\begin{cases} \varphi_t - d\varphi_{xx} \geq C\varphi, & t > 0, 0 < x < g(t), \\ B[\varphi] = 0, & t \geq 0, x = 0, \\ \varphi = 0, \quad g'(t) \geq -\mu\varphi_x, & t \geq 0, x = g(t), \end{cases}$$

then $\lim_{t \rightarrow \infty} \max_{0 \leq x \leq g(t)} \varphi(t, x) = 0$.

Applying (1.7) and Lemma 3.1, we have the following result.

Theorem 3.1 (Vanishing) *If $s_\infty < \infty$, then*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0, \quad \lim_{t \rightarrow \infty} \|v(t, \cdot)\|_{C([0, s(t)])} = 0. \quad (3.1)$$

This shows that if the two species cannot spread successfully, they will extinct in the long run.

For any given $\ell > 0$, let $\lambda_1(\ell; d_1, a)$ and $\gamma_1(\ell; d_2, b)$ be the principal eigenvalues of the T -periodic eigenvalue problems

$$\begin{cases} \phi_t - d_1\phi_{xx} - a(t, x)\phi = \lambda\phi, & 0 \leq t \leq T, 0 < x < \ell, \\ B_1[\phi](t, 0) = 0, \quad \phi(t, \ell) = 0, & 0 \leq t \leq T, \\ \phi(0, x) = \phi(T, x), & 0 \leq x \leq \ell \end{cases} \quad (3.2)$$

and

$$\begin{cases} \psi_t - d_2\psi_{xx} - b(t, x)\psi = \gamma\psi, & 0 \leq t \leq T, 0 < x < \ell, \\ B_2[\psi](t, 0) = 0, \quad \psi(t, \ell) = 0, & 0 \leq t \leq T, \\ \psi(0, x) = \psi(T, x), & 0 \leq x \leq \ell, \end{cases} \quad (3.3)$$

respectively.

Theorem 3.2 (Spreading) *Assume that (H1) holds and $s_\infty = \infty$. If the condition (2.11) is true, then we have*

$$U_*(t, x) \leq \liminf_{n \rightarrow \infty} u(t + nT, x), \quad \limsup_{n \rightarrow \infty} u(t + nT, x) \leq U^*(t, x), \quad (3.4)$$

$$V_*(t, x) \leq \liminf_{n \rightarrow \infty} v(t + nT, x), \quad \limsup_{n \rightarrow \infty} v(t + nT, x) \leq V^*(t, x) \quad (3.5)$$

uniformly in $[0, T] \times [0, L]$ for any $L > 0$, where (U^, V_*) and (U_*, V^*) are positive T -periodic solutions of (2.1) obtained in Theorem 2.1.*

Proof. The method used here is an *iterative process*. This proof not only gives the long time behavior of (u, v) but also the existence of (U^*, V_*) and (U_*, V^*) .

The proof is divided into four steps. Let functions \overline{U} , \underline{U} , \overline{V} and \underline{V} be given in the proof of Theorem 2.1. In the first two steps we shall prove, respectively, that

$$\limsup_{n \rightarrow \infty} u(t + nT, x) \leq \overline{U}(t, x) \quad \text{uniformly in } [0, T] \times [0, L] \quad (3.6)$$

and

$$\liminf_{n \rightarrow \infty} v(t + nT, x) \geq \underline{V}(t, x) \quad \text{uniformly in } [0, T] \times [0, L] \quad (3.7)$$

for any given $L > 0$. In the third one, we shall construct four sequences $\{\overline{U}_i\}$, $\{\underline{V}_i\}$, $\{\underline{U}_i\}$ and $\{\overline{V}_i\}$ satisfying

$$\underline{U} \leq \underline{U}_1 \leq \cdots \leq \underline{U}_i \leq \overline{U}_i \leq \cdots \overline{U}_1 \leq \overline{U}, \quad \underline{V} \leq \underline{V}_1 \leq \cdots \leq \underline{V}_i \leq \overline{V}_i \leq \cdots \overline{V}_1 \leq \overline{V}. \quad (3.8)$$

Proofs of (3.4) and (3.5) will be given in the last step.

Step 1. Define

$$\phi(x) = \begin{cases} u_0(x), & 0 \leq x \leq s_0, \\ 0, & x \geq s_0, \end{cases}$$

and let $w(t, x)$ be the unique positive solution of

$$\begin{cases} w_t - d_1 w_{xx} = w(a(t, x) - w), & t > 0, 0 < x < \infty, \\ B_1[w](t, 0) = 0, & t > 0, \\ w(0, x) = \phi(x), & 0 \leq x < \infty. \end{cases}$$

In view of Proposition 2.3, it follows that $\lim_{n \rightarrow \infty} w(t + nT, x) = \overline{U}(t, x)$ uniformly in $[0, T] \times [0, L]$, where $\overline{U}(t, x)$ is the unique positive solution of (2.16). On the other hand, by the comparison principle, we have $u(t, x) \leq w(t, x)$ for all $t > 0$ and $0 \leq x \leq s(t)$. Thanks to $s_\infty = \infty$, we get (3.6).

Step 2. For any $\varepsilon > 0$, denote $b_\varepsilon(t, x) = b(t, x) - h(\overline{U}(t, x) + \varepsilon(1 + x)^r)$. It follows from (2.10) and (2.17) that

$$\underline{b}_\infty - h(\overline{a}^\infty + \varepsilon) \leq \liminf_{x \rightarrow \infty} \frac{b_\varepsilon(t, x)}{x^r} \leq \limsup_{x \rightarrow \infty} \frac{b_\varepsilon(t, x)}{x^r} \leq \overline{b}^\infty - h(\underline{a}_\infty + \varepsilon)$$

uniformly in $[0, T]$. Since $\underline{b}_\infty > h\overline{a}^\infty$, there exists $\varepsilon_0 > 0$ such that $\underline{b}_\infty > h(\overline{a}^\infty + \varepsilon)$, and hence $b_\varepsilon \in \mathcal{C}_r(T)$ for all $0 < \varepsilon \leq \varepsilon_0$. For such fixed ε , by Proposition 2.2, there exists $\ell_0^\varepsilon > L$ such that $\gamma_1(\ell; d_2, b_\varepsilon) < 0$ for all $\ell \geq \ell_0^\varepsilon$.

For any fixed $0 < \varepsilon < \varepsilon_0$ and $\ell > \ell_0^\varepsilon$, capitalize on (3.6) and $s_\infty = \infty$, there exists $\tau \gg 1$ such that

$$s(t) > \ell, \quad u(t, x) < \overline{U}(t, x) + \varepsilon(1 + x)^r, \quad \forall t \geq \tau, 0 \leq x \leq \ell.$$

Consider the following auxiliary T -periodic boundary value problem

$$\begin{cases} Z_t - d_2 Z_{xx} = Z(b_\varepsilon(t, x) - Z), & 0 \leq t \leq T, 0 < x < \ell, \\ B_2[Z](t, 0) = Z(t, \ell) = 0, & 0 \leq t \leq T, \\ Z(0, x) = Z(T, x), & 0 \leq x \leq \ell. \end{cases}$$

Since $\gamma_1(\ell; d_2, b_\varepsilon) < 0$, utilizing Theorem 28.1 of [15], the above problem admits a unique positive solution, denoted by $Z_\ell^\varepsilon(t, x)$. Let $V_\ell^\varepsilon(t, x)$ be the unique positive solution of the following initial-boundary value problem

$$\begin{cases} V_t - d_2 V_{xx} = V(b_\varepsilon(t, x) - V), & t > \tau, 0 < x < \ell, \\ B_2[V](t, 0) = 0, \quad V(t, \ell) = 0, & t \geq \tau, \\ V(\tau, x) = \sigma Z_\ell^\varepsilon(\tau, x), & x \in [0, \ell], \end{cases}$$

where $0 < \sigma < 1$ is so small that $\sigma Z_\ell^\varepsilon(\tau, x) < v(\tau, x)$ in $[0, \ell]$. Obviously, the function $\chi := \sigma Z_\ell^\varepsilon$ satisfies

$$\begin{cases} \chi_t - d_2 \chi_{xx} < \chi(b_\varepsilon(t, x) - \chi), & t > \tau, 0 < x < \ell, \\ B_2[\chi](t, 0) = 0, \quad \chi(t, \ell) = 0, & t \geq \tau, \\ \chi(\tau, x) = \sigma Z_\ell^\varepsilon(\tau, x), & x \in [0, \ell]. \end{cases}$$

By the comparison principle,

$$v(t, x) \geq V_\ell^\varepsilon(t, x) \geq \chi(t, x), \quad \forall t \geq \tau, 0 \leq x \leq \ell.$$

Using the arguments of step 2 in the proof of Theorem 2.1, we can prove that $\lim_{n \rightarrow \infty} V_\ell^\varepsilon(t + nT, x) = Z_\ell^\varepsilon(t, x)$ in $C^{1,2}([0, T] \times [0, \ell])$ and $\lim_{\ell \rightarrow \infty} Z_\ell^\varepsilon(t, x) = Z^\varepsilon(t, x)$ in $C^{1,2}([0, T] \times [0, L])$, where Z^ε is the unique positive solution of T -periodic boundary value problem

$$\begin{cases} Z_t - d_2 Z_{xx} = Z(b_\varepsilon(t, x) - Z), & 0 \leq t \leq T, 0 < x < \infty, \\ B_2[Z](t, 0) = Z(t, \ell) = 0, & 0 \leq t \leq T, \\ Z(0, x) = Z(T, x), & 0 \leq x < \infty. \end{cases}$$

The existence and uniqueness of Z^ε is guaranteed by Proposition 2.3. It follows that

$$\liminf_{n \rightarrow \infty} v(t + nT, x) \geq Z^\varepsilon(t, x) \quad \text{uniformly for } (t, x) \in [0, T] \times [0, L].$$

Note that $b_\varepsilon(t, x) \rightarrow b(t, x) - h\bar{U}(t, x)$ as $\varepsilon \rightarrow 0$ and $\underline{V}(t, x)$ is the unique positive solution of (2.19), by the continuous dependence of solution with respect to parameter, we have that $\lim_{\varepsilon \rightarrow 0} Z^\varepsilon(t, x) = \underline{V}(t, x)$ uniformly in $[0, T] \times [0, L]$. Thus, (3.7) holds.

Step 3. In view of (2.10), (2.11) and (2.20), we see that $a - k\underline{V} \in \mathcal{C}_r(T)$. Same as the second step, it can be deduced that $\limsup_{n \rightarrow \infty} u(t + nT, x) \leq \bar{U}_1(t, x)$ locally uniformly for $(t, x) \in [0, T] \times [0, \infty)$, where \bar{U}_1 is the unique positive solution of

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - k\underline{V}(t, x) - U), & 0 \leq t \leq T, 0 < x < \infty, \\ B_1[U](t, 0) = 0, & 0 \leq t \leq T, \\ U(0, x) = U(T, x), & 0 \leq x < \infty. \end{cases}$$

Similarly, $\liminf_{n \rightarrow \infty} v(t + nT, x) \geq \underline{V}_1(t, x)$ locally uniformly in $[0, T] \times [0, \infty)$, where \underline{V}_1 is the unique positive solution of

$$\begin{cases} V_t - d_2 V_{xx} = V(b(t, x) - h\bar{U}_1(t, x) - V), & 0 \leq t \leq T, 0 < x < \infty, \\ B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ V(0, x) = V(T, x), & 0 \leq x < \infty. \end{cases}$$

Repeating the above procedure, we can find two sequences $\{\overline{U}_i\}$ and $\{\underline{V}_i\}$ such that

$$\limsup_{n \rightarrow \infty} u(t + nT, x) \leq \overline{U}_i(t, x), \quad \liminf_{n \rightarrow \infty} v(t + nT, x) \geq \underline{V}_i(t, x) \quad (3.9)$$

locally uniformly for $(t, x) \in [0, T] \times [0, \infty)$, here \overline{U}_i and \underline{V}_i are the unique positive solutions of

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - k\underline{V}_{i-1}(t, x) - U), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_1[U](t, 0) = 0, & 0 \leq t \leq T, \\ U(0, x) = U(T, x), & 0 \leq x < \infty \end{cases} \quad (3.10)$$

and

$$\begin{cases} V_t - d_2 V_{xx} = V(b(t, x) - h\overline{U}_i(t, x) - V), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ V(0, x) = V(T, x), & 0 \leq x < \infty, \end{cases} \quad (3.11)$$

respectively.

In the same way we can get two sequences $\{\underline{U}_i\}$ and $\{\overline{V}_i\}$ such that

$$\liminf_{n \rightarrow \infty} u(t + nT, x) \geq \underline{U}_i(t, x), \quad \limsup_{n \rightarrow \infty} v(t + nT, x) \leq \overline{V}_i(t, x) \quad (3.12)$$

locally uniformly in $[0, T] \times [0, \infty)$, here \overline{V}_i and \underline{U}_i are the unique positive solutions of

$$\begin{cases} V_t - d_2 V_{xx} = V(b(t, x) - h\underline{U}_{i-1}(t, x) - V), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ V(0, x) = V(T, x), & 0 \leq x < \infty \end{cases} \quad (3.13)$$

and

$$\begin{cases} U_t - d_1 U_{xx} = U(a(t, x) - k\overline{V}_i(t, x) - U), & 0 \leq t \leq T, \ 0 < x < \infty, \\ B_1[U](t, 0) = 0, & 0 \leq t \leq T, \\ U(0, x) = U(T, x), & 0 \leq x < \infty, \end{cases} \quad (3.14)$$

respectively.

Apply Proposition 2.4, we can show that (3.8) holds.

Step 4. Now we prove (3.4) and (3.5). Remembering (3.8), make use of the regularity theory for parabolic equations and compact argument, we assert that there exist four positive T -periodic functions $U^\infty, U_\infty, V^\infty, V_\infty \in C^{1+\frac{\epsilon}{2}, 2+\nu}([0, T] \times [0, \infty))$, such that

$$(\overline{U}_i, \underline{U}_i, \overline{V}_i, \underline{V}_i) \rightarrow (U^\infty, U_\infty, V^\infty, V_\infty) \quad \text{as } i \rightarrow \infty$$

in $C^{1,2}([0, T] \times [0, K])$ for any $K > 0$. Taking $i \rightarrow \infty$ in (3.10), (3.11), (3.13) and (3.14), it derives that both (U^∞, V_∞) and (U_∞, V^∞) are positive solutions of (2.1). Hence, by (2.12),

$$U_* \leq U_\infty \leq U^\infty \leq U^*, \quad V_* \leq V_\infty \leq V^\infty \leq V^*. \quad (3.15)$$

Arguing as step 3 in the proof of Theorem 2.1, it can be shown that any positive solution (U, V) of (2.1) must satisfy $\underline{U}_i \leq U \leq \overline{U}_i, \underline{V}_i \leq V \leq \overline{V}_i$ for all i . Thus $U_\infty \leq U \leq U^\infty$ and $V_\infty \leq V \leq V^\infty$. Since (U^*, V_*) and (U_*, V^*) are positive solutions of (2.1), we have

$$U_\infty \leq U_* \leq U^* \leq U^\infty, \quad V_\infty \leq V_* \leq V^* \leq V^\infty.$$

Recall (3.15), it yields that $U_\infty = U_*, U^* = U^\infty, V_\infty = V_*, V^* = V^\infty$. Letting $i \rightarrow \infty$ in (3.9) and (3.12), the required results (3.4) and (3.5) are obtained. \square

4 Criteria for spreading and vanishing of (1.2)

Throughout this section, we assume that (u, v, s) is the unique solution of (1.2). We first state a comparison principle.

Lemma 4.1 (*Comparison principle*) *Let $\tau > 0$, $\bar{s} \in C^1([0, \tau])$ and $\bar{s}(t) > 0$ in $[0, \tau]$. Let $\bar{u}, \bar{v} \in C(\bar{Q}) \cap C^{1,2}(Q)$ with $Q = \{(t, x) : 0 < t \leq \tau, 0 < x < \bar{s}(t)\}$. Assume that $(\bar{u}, \bar{v}, \bar{s})$ satisfies*

$$\begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} \geq \bar{u}(a(t, x) - \bar{u}), & 0 < t \leq \tau, 0 < x < \bar{s}(t), \\ \bar{v}_t - d_2 \bar{v}_{xx} \geq \bar{v}(b(t, x) - \bar{v}), & 0 < t \leq \tau, 0 < x < \bar{s}(t), \\ B_1[\bar{u}](t, 0) \geq 0, \quad B_2[\bar{v}](t, 0) \geq 0, & 0 \leq t \leq \tau, \\ \bar{u}(t, \bar{s}(t)) = \bar{v}(t, \bar{s}(t)) = 0, & 0 \leq t \leq \tau, \\ \bar{s}'(t) \geq -\mu[\bar{u}_x(t, \bar{s}(t)) + \rho \bar{v}_x(t, \bar{s}(t))], & 0 \leq t \leq \tau. \end{cases}$$

If $\bar{s}(0) \geq s_0$, $\bar{u}(0, x) \geq 0$, $\bar{v}(0, x) \geq 0$ in $[0, \bar{s}(0)]$, and $u_0(x) \leq \bar{u}(0, x)$, $v_0(x) \leq \bar{v}(0, x)$ in $[0, s_0]$, then the solution (u, v, s) of (1.2) satisfies

$$s(t) \leq \bar{s}(t) \text{ in } [0, \tau]; \quad u(t, x) \leq \bar{u}(t, x), \quad v(t, x) \leq \bar{v}(t, x) \text{ in } \bar{Q},$$

where $Q = \{(t, x) : 0 < t \leq \tau, 0 < x < s(t)\}$.

Proof. The proof is same as that of [30, Lemma 4.1] (see also the argument of [14, Lemma 5.1]), we omit the details. \square

Define

$$\mathcal{E} = \{\ell > 0 : \lambda_1(\ell; d_1, a) = 0 \text{ or } \gamma_1(\ell; d_2, b) = 0\}.$$

If one of the functions $a(t, x)$ and $b(t, x)$ satisfies the condition **(A)**, make use of Proposition 2.2, it yields that $\mathcal{E} \neq \emptyset$. Especially, when the assumption **(H1)** holds, both $a(t, x)$ and $b(t, x)$ satisfy the condition **(A)**, and hence $\mathcal{E} \neq \emptyset$.

Now we give a necessary condition of vanishing.

Lemma 4.2 *Assume that $\mathcal{E} \neq \emptyset$ and set $s^* = \min \mathcal{E}$. If $s_\infty < \infty$, then $s_\infty \leq s^*$. Hence, $s_0 \geq s^*$ implies $s_\infty = \infty$ for all $\mu > 0$.*

Proof. First of all, $s^* > 0$ since $\lambda_1(\ell; d_1, a) > 0$, $\gamma_1(\ell; d_2, b) > 0$ when $0 < \ell \ll 1$. Without loss of generality we assume that $\lambda_1(s^*; d_1, a) = 0$.

If $s_\infty > s^*$, then $\lambda_1(s_\infty; d_1, a) < 0$ since $\lambda_1(\ell; d_1, a)$ is strictly decreasing in ℓ . By the continuity of $\lambda_1(\ell; d_1, a)$, there exists $\varepsilon > 0$ such that $\lambda_1(s_\infty; d_1, a - k\varepsilon) < 0$. In view of Theorem 3.1, $\lim_{t \rightarrow \infty} \|v(t, \cdot)\|_{C([0, s(t)])} = 0$. There exists $\tau \gg 1$ such that $\lambda_1(s(\tau); d_1, a - k\varepsilon) < 0$ and $v(t, x) \leq \varepsilon$ for all $t \geq \tau$, $x \in [0, s(\tau)]$. Let w be the unique solution of

$$\begin{cases} w_t - d_1 w_{xx} = w(a(t, x) - k\varepsilon - w), & t > \tau, 0 < x < s(\tau), \\ B_1[w](t, 0) = w(t, s(\tau)) = 0, & t > \tau, \\ w(\tau, x) = u(\tau, x), & 0 \leq x \leq s(\tau). \end{cases}$$

Then $u \geq w$ in $[\tau, \infty) \times [0, s(\tau)]$ by the comparison principle. Note that $\lambda_1(s(\tau); d_1, a - k\varepsilon) < 0$, it follows from Theorem 28.1 of [15] that $w(t + nT, x) \rightarrow Z(t, x)$ as $n \rightarrow \infty$ uniformly on $[0, T] \times [0, s(\tau)]$, where $Z(t, x)$ is the unique positive solution of the following T -periodic boundary value problem

$$\begin{cases} Z_t - d_1 Z_{xx} = Z(a(t, x) - k\varepsilon - Z), & 0 \leq t \leq T, 0 < x < s(\tau), \\ B_1[Z](t, 0) = Z(t, s(\tau)) = 0, & 0 \leq t \leq T, \\ Z(0, x) = Z(T, x), & 0 \leq x \leq s(\tau). \end{cases}$$

Since $u \geq w$ in $[\tau, \infty) \times [0, s(\tau)]$, we immediately obtain

$$\liminf_{n \rightarrow \infty} u(t + nT, x) \geq Z(t, x), \quad \forall (t, x) \in [0, T] \times [0, s(\tau)].$$

This is a contradiction with the first formula of (3.1). The proof is complete. \square

In the following, with the parameter s_0 satisfying $s_0 < s^*$ and (u_0, v_0) being fixed, let us discuss the effect of the coefficient μ on the spreading and vanishing. We first give a lemma.

Lemma 4.3 *Let $d, C > 0$ be fixed constants and the boundary operator B be as above. For any given constants $s_0, \Lambda > 0$, and any function $\bar{u}_0 \in C^2([0, s_0])$ satisfying $B[\bar{u}_0](0) = \bar{u}_0(s_0) = 0$ and $\bar{u}_0 > 0$ in $(0, s_0)$, there exists $\mu^0 > 0$ such that when $\mu \geq \mu^0$ and (\bar{u}, \bar{s}) satisfies*

$$\begin{cases} \bar{u}_t - d\bar{u}_{xx} \geq -C\bar{u}, & t > 0, 0 < x < \bar{s}(t), \\ B[\bar{u}](t, 0) = 0 = \bar{u}(t, \bar{s}(t)), & t \geq 0, \\ \bar{s}'(t) = -\mu\bar{u}_x(t, \bar{s}(t)), & t \geq 0, \\ \bar{s}(0) = s_0, \bar{u}(0, x) = \bar{u}_0(x), & 0 \leq x \leq s_0, \end{cases}$$

we must have $\liminf_{t \rightarrow \infty} \bar{s}(t) > \Lambda$.

The proof of Lemma 4.3 is essentially similar to that of Lemma 3.2 in [36] and is hence omitted.

Recalling the estimate (1.6), as a consequence of Lemmas 4.2 and 4.3, we have

Corollary 4.1 *Assume that $\mathcal{E} \neq \emptyset$ and set $s^* = \min \mathcal{E}$. If $s_0 < s^*$, then there exists $\mu^0 > 0$ depending on (u_0, v_0, s_0) such that $s_\infty = \infty$ if $\mu \geq \mu^0$.*

Lemma 4.4 *Assume that $\mathcal{E} \neq \emptyset$ and set $s^* = \min \mathcal{E}$. If $s_0 < s^*$, then there exists $\mu_0 > 0$, such that $s_\infty \leq s^*$ for all $\mu \leq \mu_0$.*

Proof. This proof is similar to that of Lemma 5.2 in [33]. Here we give the sketch for completeness and readers' convenience. Since $s_0 < s^* = \min \mathcal{E}$, we have that $\lambda_1(s_0; d_1, a) > 0$ and $\gamma_1(s_0; d_2, b) > 0$.

Let $\phi(t, x)$ and $\psi(t, x)$ be, respectively, the positive eigenfunctions corresponding to $\lambda_1 := \lambda_1(s_0; d_1, a)$ and $\gamma_1 := \gamma_1(s_0; d_2, b)$ of (3.2) and (3.3) with $\ell = s_0$. The following conclusions are obvious:

- (i) $\phi_x(t, s_0) < 0, \psi_x(t, s_0) < 0$ in $[0, T]$;
- (ii) $\phi(t, 0) > 0$ in $[0, T]$ when $\beta_1 > 0, \psi(t, 0) > 0$ in $[0, T]$ when $\beta_2 > 0$;
- (iii) $\phi_x(t, 0) > 0$ in $[0, T]$ when $\beta_1 = 0$, and $\psi_x(t, 0) > 0$ in $[0, T]$ when $\beta_2 = 0$.

In view of the above facts (i)-(iii) and the regularity of ϕ and ψ , we know that there exists a constant $C > 0$ such that

$$x\phi_x(t, x) \leq C\phi(t, x), \quad x\psi_x(t, x) \leq C\psi(t, x), \quad \forall (t, x) \in [0, T] \times [0, s_0]. \quad (4.1)$$

Let $0 < \delta, \sigma < 1$ and $\Lambda > 0$ be constants, which will be determined later. Set

$$g(t) = 1 + 2\delta - \delta e^{-\sigma t}, \quad \xi(t) = \int_0^t g^{-2}(\tau) d\tau, \quad t \geq 0,$$

$$w(t, x) = \Lambda e^{-\sigma t} \phi(\xi(t), y), \quad z(t, x) = \Lambda e^{-\sigma t} \psi(\xi(t), y), \quad y = \frac{x}{g(t)}, \quad 0 \leq x \leq s_0 g(t).$$

Firstly, for any given $0 < \varepsilon \ll 1$, since a and b are uniformly continuous in $[0, T] \times [0, 3s_0]$ and T -periodic in t , there exists $0 < \delta_0(\varepsilon) \ll 1$ such that, for all $0 < \delta \leq \delta_0(\varepsilon)$ and $0 < \sigma < 1$,

$$|a(\xi(t), y(t, x)) - g^2(t)a(t, x)| \leq \varepsilon, \quad |b(\xi(t), y(t, x)) - g^2(t)b(t, x)| \leq \varepsilon \quad (4.2)$$

for $t \geq 0$ and $0 \leq x \leq s_0 g(t)$. Remembering $\lambda_1, \gamma_1 > 0$, in view of (4.1) and (4.2), the direct calculation yields that

$$w_t - d_1 w_{xx} - w(a(t, x) - w) > v(-\sigma - \varepsilon - \sigma C + \lambda_1/4) > 0, \quad (4.3)$$

$$z_t - d_2 z_{xx} - z(b(t, x) - z) > z(-\sigma - \varepsilon - \sigma C + \gamma_1/4) > 0 \quad (4.4)$$

for all $t > 0$ and $0 < x < s_0 g(t)$ provided $0 < \sigma, \varepsilon \ll 1$. Carefully analysis gives

$$B_1[w](t, 0) \geq 0, \quad B_2[z](t, 0) \geq 0, \quad w(t, s_0 g(t)) = 0, \quad z(t, s_0 g(t)) = 0, \quad \forall t > 0. \quad (4.5)$$

Fix $0 < \sigma, \varepsilon \ll 1$ and $0 < \delta \leq \delta_0(\varepsilon)$. Based on the regularities of $u_0(x)$, $v_0(x)$, $\phi(0, x)$ and $\psi(0, x)$, one can choose $\Lambda \gg 1$ such that

$$u_0(x) \leq \Lambda \phi\left(0, \frac{x}{1+\delta}\right) = w(0, x), \quad v_0(x) \leq \Lambda \psi\left(0, \frac{x}{1+\delta}\right) = z(0, x), \quad \forall 0 \leq x \leq s_0. \quad (4.6)$$

According to $s_0 g'(t) = s_0 \sigma \delta e^{-\sigma t}$ and

$$w_x(t, s_0 g(t)) = \frac{1}{g(t)} \Lambda e^{-\sigma t} \phi_y(\xi(t), s_0), \quad z_x(t, s_0 g(t)) = \frac{1}{g(t)} \Lambda e^{-\sigma t} \psi_y(\xi(t), s_0),$$

there exists $\mu_0 > 0$ such that

$$s_0 g'(t) \geq -\mu(w_x(t, s_0 g(t)) + \rho z_x(t, s_0 g(t))), \quad \forall 0 < \mu \leq \mu_0, \quad t \geq 0. \quad (4.7)$$

Remembering (4.3)-(4.7), we can apply the comparison principle (Lemma 4.1) to $(u, v, s(t))$ and $(w, z, s_0 g(t))$, and then derive that

$$s(t) \leq s_0 g(t), \quad u(t, x) \leq w(t, x), \quad v(t, x) \leq z(t, x), \quad \forall t \geq 0, \quad 0 \leq x \leq s(t).$$

Hence $s_\infty \leq s_0 g(\infty) = s_0(1 + 2\delta)$ for all $0 < \mu \leq \mu_0$. The proof is complete. \square

Now, let us give the criteria for spreading and vanishing of the problem (1.2).

Theorem 4.1 *Assume that $\mathcal{E} \neq \emptyset$ and set $s^* = \min \mathcal{E}$.*

- (i) *If $s_0 \geq s^*$, then $s_\infty = \infty$ for all $\mu > 0$;*
- (ii) *If $s_0 < s^*$, then there exist $\mu^* \geq \mu_* > 0$, depending on (u_0, v_0, s_0) , such that $s_\infty \leq s^*$ if $\mu \leq \mu_*$, and $s_\infty = \infty$ if $\mu > \mu^*$.*

Proof. Remember Lemmas 4.2 and 4.4 and Corollary 4.1, the proof is similar to that of Theorem 5.2 in [30]. We omit the details. \square

We have mentioned in the above that if one of the functions $a(t, x)$ and $b(t, x)$ satisfies the condition **(A)**, then $\mathcal{E} \neq \emptyset$.

5 The problem (1.3)

In this section, we briefly discuss the problem (1.3). So, (u, v, s) means the unique solution of (1.3) throughout this section.

Theorem 5.1 (*Vanishing*) *Assume that the condition (H1) holds. If $s_\infty < \infty$, then*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0, \quad (5.1)$$

$$\lim_{n \rightarrow \infty} v(t + nT, x) = V(t, x) \quad \text{uniformly in } [0, T] \times [0, L] \quad (5.2)$$

for any $L > 0$, where V is the unique positive solution of the following T -periodic boundary value problem

$$\begin{cases} V_t - d_2 V_{xx} = V(b(t, x) - V), & 0 \leq t \leq T, \quad 0 < x < \infty, \\ B_2[V](t, 0) = 0, & 0 \leq t \leq T, \\ V(0, x) = V(T, x), & 0 \leq x < \infty. \end{cases} \quad (5.3)$$

This shows that if the invasive species u cannot spread successfully, it will extinct in the long run.

Proof. The proof of (5.1) is the same as that of (3.1). The existence and uniqueness of $V(t, x)$ is guaranteed by Proposition 2.3. Thanks to (5.1), similarly to the proof of Theorem 3.2, we can show that $\liminf_{n \rightarrow \infty} v(t + nT, x) \geq V(t, x)$ and $\limsup_{n \rightarrow \infty} v(t + nT, x) \leq V(t, x)$ uniformly in $[0, T] \times [0, L]$. This finishes the proof. \square

When $s_\infty = \infty$, Theorem 3.2 (Spreading) remains hold.

In the following three lemmas, we assume that the function $a(t, x)$ satisfies condition (A) and take $s_* > 0$ such that $\lambda_1(s_*, d_1, a) = 0$. Obviously, s_* exists uniquely.

Lemma 5.1 *If $s_\infty < \infty$, then $s_\infty \leq s_*$. Hence, $s_0 \geq s_*$ implies $s_\infty = \infty$ for all $\mu > 0$.*

The proof of Lemma 5.1 is essentially same as that of Lemma 4.2 and is hence omitted here.

Lemma 5.2 *If $s_0 < s_*$, then there exist $0 < \mu_0 < \mu^0$, such that $s_\infty = \infty$ for $\mu \geq \mu^0$, and $s_\infty \leq s_*$ for $\mu \leq \mu_0$.*

Proof. Recalling the estimates obtained in Theorem 1.2, and applying Lemmas 4.3 and 5.1, we can derive that there exists $\mu^0 > 0$, such that $s_\infty = \infty$ for $\mu \geq \mu^0$. On the other hand, since u satisfies

$$\begin{cases} u_t - d_1 u_{xx} < u(a(t, x) - u), & t > 0, \quad 0 < x < s(t), \\ B_1[u] = 0, & t > 0, \quad = 0, \\ u = 0, \quad s'(t) = -\mu u_x, & t > 0, \quad x = s(t), \\ s(0) = s_0, \quad u(0, x) = u_0(x), & 0 \leq x \leq s_0, \end{cases}$$

by the known results for the logistic equation (see Lemma 5.2 of [33]) and comparison principle, we can show that there exists $\mu_0 > 0$, such that $s_\infty \leq s_*$ for $\mu \leq \mu_0$. The proof is complete. \square

In the same way as the proof of Lemma 2.6 in [10], it can be proved that (u, v, s) is monotone increasing in μ . Similarly to the proof of Lemma 4.9 in [10], we have the following lemma.

Lemma 5.3 *If $s_0 < s_*$, then there exist $\mu^* > 0$, such that $s_\infty = \infty$ for $\mu > \mu^*$, while $s_\infty \leq s_*$ for $\mu \leq \mu^*$.*

It is worth mentioning that the assumption **(H1)** implies condition **(A)**. Summarizing the above conclusions we obtain the following spreading-vanishing dichotomy and sharp criteria for spreading and vanishing.

Theorem 5.2 *Under the condition **(H1)**, we have the following alternative conclusion:*

Either

(i) *spreading: $s_\infty = \infty$, both (3.4) and (3.5) hold,*

or

(ii) *vanishing: $s_\infty \leq s_*$, and $\lim_{t \rightarrow \infty} \|u(t, \cdot)\|_{C([0, s(t)])} = 0$, $\lim_{n \rightarrow \infty} v(t + nT, x) = V(t, x)$ uniformly in $[0, T] \times [0, L]$ for any given $L > 0$, where V is the unique positive T -periodic solution of (5.3).*

Moreover,

(iii) *If $s_0 \geq s_*$, then $s_\infty = \infty$ for all $\mu > 0$.*

(iv) *If $s_0 < s_*$, then there exist $\mu^* > 0$, such that $s_\infty = \infty$ for $\mu > \mu^*$, while $s_\infty \leq s_*$ for $\mu \leq \mu^*$.*

Finally, we estimate the asymptotic spreading speed of the free boundary $s(t)$ when spreading occurs. To this aim, let us first state a known result, which plays an important role in the study of asymptotic spreading speed. For a T -periodic function $f(t)$, we define

$$\bar{f} = \frac{1}{T} \int_0^T f(t) dt.$$

Proposition 5.1 ([7, Section 2]) *Let $d > 0$ and $0 < \nu < 1$ be the given constants. Assume that $F, \varphi \in C^\nu([0, T])$ are T -periodic functions, φ is positive and F is nonnegative in $[0, T]$. Then the problem*

$$\begin{cases} w_t - dw_{xx} + F(t)w_x = \varphi(t)w - w^2, & 0 \leq t \leq T, 0 < x < \infty, \\ w(t, 0) = 0, & 0 \leq t \leq T, \\ w(0, x) = w(T, x), & 0 \leq x < \infty \end{cases}$$

has a positive T -periodic solution $w^F(t, x) \in C^{1,2}([0, T] \times [0, \infty))$ if and only if $\bar{F} < 2\sqrt{d\bar{\varphi}}$, and such a solution is unique when it exists. Moreover, the following hold:

(i) *$w_x^F(t, x) > 0$ and $w^F(t, x) \rightarrow z(t)$ uniformly in $[0, T]$ as $x \rightarrow \infty$, where $z(t)$ is the unique positive periodic solution of the problem*

$$z' = \varphi(t)z - z^2, \quad 0 \leq t \leq T; \quad z(0) = z(T);$$

(ii) *For any given nonnegative T -periodic function $G \in C^\nu([0, T])$ satisfying $\bar{G} < 2\sqrt{d\bar{\varphi}}$, the assumption $G \leq, \neq F$ implies $w_x^G(t, 0) > w_x^F(t, 0)$, $w^G(t, x) > w^F(t, x)$ for $0 \leq t \leq T$ and $x > 0$;*

(iii) *For each $\mu > 0$, there exists a unique positive T -periodic function $F_0(t) = F_0(d, \mu, \varphi)(t) \in C^\nu([0, T])$ such that $\mu w_x^{F_0}(t, 0) = F_0(t)$ in $[0, T]$, and $0 < \bar{F}_0 < 2\sqrt{\bar{\varphi}d}$.*

Theorem 5.3 *Under the condition **(H1)** with $r = 0$ we further assume that (2.11) hold. When the spreading occurs, i.e., $s_\infty = \infty$, we have*

$$\limsup_{t \rightarrow \infty} \frac{s(t)}{t} \leq \frac{1}{T} \int_0^T F_0(d_1, \mu, a^\infty - kz_1)(t) dt, \quad \liminf_{t \rightarrow \infty} \frac{s(t)}{t} \geq \frac{1}{T} \int_0^T F_0(d_1, \mu, a_\infty - kz_2)(t) dt,$$

where $z_1(t)$ and $z_2(t)$ are the unique positive solutions of (2.14) and (2.15), respectively.

Proof. This proof is similar to that of Theorem 4.4 in [7] with some modifications. Here we give the sketch for completeness and readers' convenience.

First of all, in view of (2.14) and (2.15), it is easy to see that

$$z_1(t) \leq \max_{[0,T]} b_\infty(t) \leq \bar{b}^\infty, \quad z_2(t) \leq \max_{[0,T]} b^\infty(t) = \bar{b}^\infty, \quad \forall t \in [0, T].$$

Therefore,

$$a^\infty(t) - kz_1(t) \geq \underline{a}_\infty - k\bar{b}^\infty > 0, \quad a_\infty(t) - kz_2(t) \geq \underline{a}_\infty - k\bar{b}^\infty > 0, \quad \forall t \in [0, T]$$

by the condition (2.11).

Step 1. Let (U^*, V_*) and (U_*, V^*) be positive solutions of (2.1) obtained in Theorem 2.1. Apply the last two conclusions of (2.13), it yields that

$$z_1(t) \leq \liminf_{x \rightarrow \infty} V_*(t, x), \quad \limsup_{x \rightarrow \infty} V^*(t, x) \leq z_2(t)$$

uniformly in $[0, T]$. By the same way as that of step 1 in the proof of [7, Theorem 4.4], we can show that

$$\limsup_{x \rightarrow \infty} U^*(t, x) \leq \bar{\psi}(t), \quad \liminf_{x \rightarrow \infty} U_*(t, x) \geq \underline{\psi}(t) \quad \text{uniformly in } [0, T], \quad (5.4)$$

where $\bar{\psi}(t)$ and $\underline{\psi}(t)$ are, respectively, the unique positive T -periodic solutions of

$$\bar{\psi}'(t) = \bar{\psi}(a^\infty(t) - kz_1(t) - \bar{\psi}), \quad \bar{\psi}(0) = \bar{\psi}(T)$$

and

$$\underline{\psi}'(t) = \underline{\psi}(a_\infty(t) - kz_2(t) - \underline{\psi}), \quad \underline{\psi}(0) = \underline{\psi}(T).$$

Step 2. For the given $0 < \varepsilon \ll 1$, by (5.4), there exists $\ell^* = \ell^*(\varepsilon) \gg 1$ such that

$$U^*(t, x) \leq \bar{\psi}_\varepsilon(t), \quad U_*(t, x) \geq \underline{\psi}_\varepsilon(t) \quad \text{in } [0, T] \times [\ell^*, \infty),$$

where $\bar{\psi}_\varepsilon(t)$ and $\underline{\psi}_\varepsilon(t)$ are, respectively, the unique positive T -periodic solutions of

$$\bar{\psi}_\varepsilon'(t) = \bar{\psi}_\varepsilon(a^\infty(t) - kz_1(t) + \varepsilon - \bar{\psi}_\varepsilon), \quad \bar{\psi}_\varepsilon(0) = \bar{\psi}_\varepsilon(T)$$

and

$$\underline{\psi}_\varepsilon'(t) = \underline{\psi}_\varepsilon(a_\infty(t) - kz_2(t) - \varepsilon - \underline{\psi}_\varepsilon), \quad \underline{\psi}_\varepsilon(0) = \underline{\psi}_\varepsilon(T).$$

Because Theorem 3.2 holds for the problem (1.3), in view of (3.4), (3.5) and $s_\infty = \infty$, we have that there exists a positive integer $n = n(\ell^*)$ such that $s(nT) > 3\ell^*$ and

$$\underline{\psi}_{2\varepsilon}(t) \leq u(t + nT, 2\ell^*) \leq \bar{\psi}_{2\varepsilon}(t), \quad \forall t \geq 0.$$

Follow the arguments of steps 2 and 3 in the proof of [7, Theorem 4.4] step by step, we can obtain the desired results. The details are omitted here. \square

Remark 5.1 *The main difference between (1.2) and (1.3) is the following. For the problem (1.2), the criteria we got for spreading and vanishing are not sharp, due to the lack of monotonicity of solution in μ . Also, the estimate of the asymptotic spreading speed of free boundary has not been obtained for the problem (1.2).*

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